

The Isoperimetric Inequality and Associated Boundary Problems

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1. Introduction. For a simple closed plane curve Γ of length L and enclosed area A there have been many proofs of the isoperimetric inequality $4\pi A \leq L^2$, with the sign of equality holding if and only if Γ is a circle. In particular (see [5], pp. 185–187), one of the most elementary proofs uses the so-called “Wirtinger inequality”, which expresses the extremal property of the second proper value of the boundary problem

$$(1.1) \quad \eta'' + \lambda\eta = 0, \quad \eta(0) - \eta(2\pi) = 0, \quad \eta'(0) - \eta'(2\pi) = 0.$$

In Section 2 of the present paper there is presented an alternative elementary proof of this isoperimetric inequality, based on the extremal property of the smallest proper value of the boundary problem

$$(1.2) \quad \eta'' + \lambda\eta = 0, \quad \eta(0) = 0, \quad \eta(L) = 0.$$

In Section 3 it is shown that the extremal property of the smallest proper value of a boundary problem related to (1.2) affords a similar elementary proof of the modified isoperimetric inequality for the simplest instance of a plane variational problem of “Dido type”. Finally, with the aid of the extremal property of the smallest proper value of (1.2) there is established in Section 4 an integral inequality for simply connected pieces of regular surfaces, which provides an elementary proof of the result of CARLEMAN [4] (see also RADÓ [7; p. 45] and BECKENBACH & RADÓ [1]) that the isoperimetric inequality $4\pi A \leq L^2$ holds for a simply connected rectifiable piece of a regular minimal surface.

2. The classical isoperimetric inequality. Suppose that

$$(2.1) \quad \Gamma: x = x(s), \quad y = y(s), \quad 0 \leq s \leq L,$$

is a simple closed rectifiable curve in the xy -plane, where the parameter s denotes

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