

Positive Operators

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The classical Frobenius theorems concerning matrices with non-negative elements have in recent years proved of fundamental importance and aid in the study of vibrations of mechanical systems [8], stochastic processes [7], and mathematical economics [5]. The first Frobenius theorem states that if $A = (a_{ij})$, $i, j = 1, \dots, n$, is a square non-singular matrix with non-negative elements, then among the eigenvalues of this matrix there exists a positive number ρ to which corresponds a non-zero eigenvector $v = (v_1, v_2, \dots, v_n)$ such that $Av = \rho v$ and $v_i \geq 0$. Furthermore, the magnitudes of all other eigenvalues of A do not exceed the value ρ .

One of the principal components of the analysis of this theorem revolves about the concept that a non-negative matrix may be regarded as a linear mapping which transforms the positive orthant into itself. The positivity preserving feature of the mapping can be exploited in a variety of ways. The newest trick is to use suitable contraction properties of projective mappings related to the matrix A [2], [14]. In this paper, we appeal to complex variable theorems about power series with positive coefficients. These appear to be easily susceptible to generalization (see Section 4 and Section 9).

In recent years numerous types of generalizations of the Frobenius theorem have been advanced. First, the positive orthant can be replaced by a general convex cone and we speak of linear transformations leaving the cone invariant. We shall still refer to such transformations as positive operators. Second, the extensions of the Frobenius theorem to infinite dimensional situations can be accomplished in varying degrees of generality. Some of the notable contributors in this respect include JENTZSCH, KAKUTANI, BIRKHOFF [2], BONNALL [4], KREIN & GANTMACHER [8], and KREIN & RUTMAN [12] (henceforth abbreviated K-R). (A thorough bibliography covering the years prior to 1948 is included in [12].)

The last reference embodies a comprehensive synthesis covering wide ramifications of the theory of positive mappings. This work serves as the chief inspiration for the studies in this paper and its influence on us will be abundantly manifest. We would also like to express our great indebtedness to H. F. BOHNENBLUST for our many valuable discussions which took place in the course