

# *Integral Formulas for Hypersurfaces in Euclidean Space and Their Applica- tions to Uniqueness Theorems*

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**Introduction.** Let  $E$  be the Euclidean space of dimension  $n + 1$ . By a hypersurface in  $E$  we mean a differentiable manifold  $M$  of dimension  $n$  and a differentiable mapping  $x: M \rightarrow E$ , whose functional matrix is everywhere of rank  $n$ . Since the strength of the differentiability assumptions will not be the issue, we suppose our manifolds and mappings to be of class  $C^\infty$ .

If  $M$  is compact, certain immediate integral formulas are valid. More generally, we will derive integral formulas for the hypersurface  $x$  and a second hypersurface  $x': M \rightarrow E$ . Such formulas are generalizations of well-known formulas in the theory of convex bodies, which express the mixed volumes of two convex bodies as integrals [1].

Given two compact hypersurfaces, a rigidity or uniqueness theorem gives a sufficient condition such that commutativity holds in the diagram

$$\begin{array}{ccc} M & \xrightarrow{x} & E \\ & \searrow_{x'} & \swarrow_{T} \\ & & E \end{array}$$

where  $T$  is a motion in  $E$ . Of interest in differential geometry are conditions expressed in terms of the relative curvature of the hypersurface. We review its definition as follows:

‡ Suppose  $M$  to be oriented. Then to  $p \in M$  there is a uniquely determined unit normal vector  $\xi(p)$  at  $x(p)$ . We put

$$(1) \quad \text{I} = dx^2, \quad \text{II} = d\xi \, dx, \quad \text{III} = d\xi^2,$$

where  $dx, d\xi$  are vector-valued linear differential forms in  $M$  and multiplication is in the sense of the scalar product in  $E$ . These are three quadratic differential forms in  $M$  (the "fundamental forms"), of which I is positive definite. The eigenvalues  $k_1, \dots, k_n$  of II relative to I are called the principal curvatures. If the Gauss-Kronecker curvature  $K = k_1 \cdots k_n \neq 0$ , the reciprocals  $1/k_1, \dots, 1/k_n$  are called the radii of principal curvature; they are the eigenvalues of II

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