

The Stefan Problem for a Nonlinear Equation

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1. Introduction. The Stefan problem is the problem of solving the diffusion or heat equation for a function c in a region bounded in part by an unknown moving surface which must also be determined. Such a problem is also called a free boundary problem. Problems of this type arise in many phase change processes, such as the freezing of a liquid. Then the interface between the two phases (*e.g.* liquid and solid) is the unknown or free boundary which must be found. STEFAN [1] solved explicitly a special problem of this kind in which the heat conductivity or diffusion coefficient and the other data of the problem were constants. In his case the solution depended only upon the time t and one Cartesian coordinate x , and the interface was a plane. It turned out that the displacement of the interface was proportional to $t^{\frac{1}{2}}$ and that c was a function of the product $xt^{-\frac{1}{2}}$. These two conclusions are consequences of dimensional analysis. C. ZENER [2] has solved explicitly the corresponding two and three-dimensional problems in which x denotes radial distance from an axis or radial distance from a point. In these cases the interface is a circular cylinder or sphere which initially has zero radius. The radius grows proportionally to $t^{\frac{1}{2}}$ and c depends only upon $xt^{-\frac{1}{2}}$, as dimensional analysis again requires.

In this paper we consider a one-dimensional problem like that solved by STEFAN. However, we permit the heat conductivity or diffusion coefficient to depend upon the solution c . Then the diffusion equation becomes nonlinear, in contrast to the linear equation treated by STEFAN. Dimensional analysis still requires the displacement of the interface to be proportional to $t^{\frac{1}{2}}$ and the solution c to be a function of $xt^{-\frac{1}{2}}$. We prove that there does exist a solution of the problem satisfying these requirements, provided that the diffusion coefficient lies between positive bounds and has a continuous derivative. In addition, a restriction must be imposed upon one constant in the problem. This result was originally conjectured by H. FRISCH and proved by W. L. MIRANKER [3] in a rather more complicated manner than that given here.

It has been shown by W. T. KYNER [4] that the present result is also true for the two and three-dimensional cases analogous to those treated by ZENER. KYNER's proof is practically the same as the present one. J. MORRISON [5] has proved a comparison theorem for solutions of the present problem from which