

A Uniqueness Theorem on Closed Convex Hypersurfaces in Euclidean Space

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The object of this note is to prove the following theorem:

Let Σ, Σ' be two closed, strictly convex, C^2 -differentiable hypersurfaces in a Euclidean space of dimension $n + 1$ (≥ 3). Let $f: \Sigma \rightarrow \Sigma'$ be a diffeomorphism such that Σ and Σ' have parallel outward normals at $p \in \Sigma$ and $p' = f(p)$ respectively. Denote by $P_l(p)$ (respectively $P_l(p')$) the l^{th} elementary symmetric function of the principal radii of curvature of Σ (resp. Σ') at p (resp. p'). If, for a fixed l , $2 \leq l \leq n$, we have

$$(1) \quad P_{l-1}(p) \leq P_{l-1}(p'), \quad P_l(p) \geq P_l(p')$$

for all points $p \in \Sigma$, then f is a translation.

The interest in this theorem lies in the fact that the conditions (1) involve only inequalities. The proof of the theorem depends on some integral formulas established in a previous paper of one of us² and on an algebraic inequality.

Let $(\lambda_{ik}), (\lambda'_{ik}), 1 \leq i, k \leq n$, be positive definite symmetric matrices. We define the polynomials $P_{rs}(\lambda, \lambda')$ by means of the equation

$$(2) \quad \det(\delta_{ik} + y\lambda_{ik} + y'\lambda'_{ik}) = \sum_{0 \leq r, s \leq n} \frac{n!}{r! s! (n - r - s)!} P_{rs}(\lambda, \lambda') y^r y'^s.$$

$P_{rs}(\lambda, \lambda')$ are thus homogeneous polynomials of degrees r, s in $\lambda_{ik}, \lambda'_{ik}$, respectively. It is easy to derive from the definition that

$$(3) \quad rP_{r-1,1}(\lambda, \lambda') = \sum_{i \leq k} \lambda'_{ik} \frac{\partial P_{r0}}{\partial \lambda_{ik}}.$$

Since $P_{r0}(\lambda, \lambda')$ is independent of λ'_{ik} , we shall write $P_{r0}(\lambda, \lambda') = P_r(\lambda)$; then $P_{0r}(\lambda, \lambda') = P_r(\lambda')$.

Lemma. Let $\alpha = (\alpha_{ik}), \lambda = (\lambda_{ik})$ be positive definite symmetric matrices such that, for a fixed l , $2 \leq l \leq n$,

$$(4) \quad P_{l-1}(\alpha) \leq P_{l-1}(\lambda), \quad P_l(\alpha) \geq P_l(\lambda).$$

¹The first two authors are partially supported by the National Science Foundation, and the third author is supported by the Air Force Office of Scientific Research.

²S. S. CHERN, *Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems*, this Journal, **8** (1959), pp. 947-955. This paper will be quoted as IFH.