

# Polynomials Associated with Divisors\*

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**Introduction.** Let  $X$  be an irreducible, normal, projective variety of dimension  $r$  with sheaf of local rings  $L$ , and defined over an arbitrary groundfield. We select  $n$ , *not necessarily distinct*, divisors  $D_1, \dots, D_n$  of  $X$  which are all locally linearly equivalent to zero, and we select an algebraic coherent sheaf  $F$  of  $X$ . We proved in Theorem 9.1 of [3] (square brackets refer to the references) that the Euler-Poincaré characteristic  $\chi(X, F(x_1 D_1 + \dots + x_n D_n))$  is a rational polynomial in  $x_1, \dots, x_n$  for all  $x_1, \dots, x_n \in \mathbb{Z}$ . The purpose of the present paper is to investigate the coefficients of this polynomial. (*Rational polynomial* means a polynomial with rational numbers as coefficients.)

Let  $Q$  be the field of rational numbers. We select  $n$  nonzero elements  $h_1, \dots, h_n \in Q$ , define the polynomials  $C(x_1, i_1, h_1), \dots, C(x_n, i_n, h_n)$  as in Definition 1.1, and denote the product of these  $n$  rational polynomials by  $[i_1, \dots, i_n]$ . The polynomials  $[i_1, \dots, i_n]$ , for  $i_1, \dots, i_n \geq 0$ , form a linearly independent  $Q$ -basis of  $Q[x_1, \dots, x_n]$ , *i.e.* if  $\alpha \in Q[x_1, \dots, x_n]$ , then  $\alpha = \sum a(i_1, \dots, i_n)[i_1, \dots, i_n]$ , where the rational numbers  $a(i_1, \dots, i_n)$  are uniquely determined by  $\alpha$ . The rational polynomials  $\alpha$  we are concerned with evidently have the special property that they take on integral values for integral values of the variables. We conclude easily from Chapter II of [1] that, for polynomials with this property, all  $a(i_1, \dots, i_n)$  become integers if each "increment"  $h_i$  is chosen equal to  $+1$  or  $-1$ .

If  $F = L$  and  $n = 1$  and  $D$  is a divisor cut out on  $X$  by a hyperplane of the ambient projective space, the polynomial  $\chi(X, L(xD))$  is the classical Hilbert polynomial of  $X$ . It is customary to expand this polynomial in terms of the "binomial coefficients"  $\binom{x}{0} = 1, \binom{x}{1}, \dots, \binom{x}{r}$ . Precisely then, it is customary to choose the increment  $h = 1$ , and to put  $\chi(X, L(xD)) = \sum a(i)[i]$ , where the summation index  $i$  runs from 0 to  $r$ . (Observe that now  $[i] = C(x, i, 1) = \binom{x}{i}$ .)

The result is unsatisfactory because only the coefficients  $a(0)$  and  $a(r)$  have good geometric meaning. We have observed that if we use the increment  $h = -1$  instead of  $h = +1$ ,  $a(0)$  and  $a(r)$  remain unchanged, but all the intermediate

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