

Boxing Inequalities

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Introduction. In his measure theoretic investigations on surface area in n dimensional coordinate space W. H. FLEMING encountered, then posed, the following question: Is areal measure generated by polyhedral covers equivalent, in the sense of having the same null sets, to that generated by spherical covers? In order to attack this problem I shall first recast it in a combinatorially more amenable form.

Areal measure generated by spheres is clearly equivalent to that generated by coordinate cubes, that is, cubes all of whose faces lie in coordinate planes, and, indeed, to that by rational coordinate cubes, henceforth called merely cubes.

Moreover, it is not hard to see that areal measure generated by polyhedra is equivalent to that generated by coordinate polyhedra, that is, polyhedra all of whose faces lie in coordinate planes, and, indeed, to that by rational coordinate polyhedra, henceforth called polycubes.

Thus we are led to reformulate FLEMING's problem as follows: Are cubic and polycubic areal measures equivalent? I shall show that they are.

This result is proved by means of the following cubic combinatorial covering inequality—the boxing inequality: There is a finite constant b , depending only on the dimension n , such that any polycube P can be covered by finitely many nonoverlapping cubes Q whose total area $\sum \sigma Q$ is not too large compared with the area σP of P in the sense that

$$\sum \sigma Q \leq b \sigma P.$$

Cubes and polycubes. Let n dimensional real coordinate space R^n be endowed with the cubic metric whereby the distance between two points is their maximum absolute coordinate difference. Denote the diameter of a set X in this space by $|X|$ and the boundary of X by ∂X .

A (closed rational coordinate) cube Q is, in the cubic metric, a sphere with rational center and diameter. Its diameter $|Q|$ is its edge length, so its volume

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