

The Exterior Dirichlet Problem for Second Order Elliptic Partial Differential Equations

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In this paper we shall treat the exterior Dirichlet problem for linear elliptic equations of second order in $n \geq 2$ independent variables. Our main concern will be with the existence and uniqueness of solutions, and with the striking effect which the asymptotic behavior of the coefficients has on the results.

It will be convenient to preface our considerations with some remarks concerning the special case of the Laplace equation. In treating the exterior Dirichlet problem for that equation it is clear that some restriction on the behavior of the solution at distant points of space is necessary in order to insure unicity. A primary condition is that the solution be bounded. Indeed, for the case $n = 2$ this condition alone suffices for uniqueness, as is well known. On the other hand, in three or more dimensions there are an infinity of bounded solutions corresponding to any given boundary data, so that some additional restriction must be imposed. Since every bounded harmonic function defined in an exterior region tends to a limit at infinity, it is natural to assign this limit value as the specifying condition. The exterior problem then becomes uniquely solvable, and, moreover, as the limit value ranges over the real numbers, every bounded solution satisfying the given data is obtained.

Consider now the general elliptic equation of second order, which we write in the form

$$(1) \quad Lu = a_{ik}(x)u_{ik} + b_i(x)u_i + c(x)u = f(x),$$

where $x = (x_1, \dots, x_n)$ and

$$u_i = \frac{\partial u}{\partial x_i}, \quad a_{ik}u_{ik} = \sum_1^n a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k}, \quad \text{etc.}$$

By the exterior Dirichlet problem we mean the following: Let \mathcal{E} be a given open region in the Euclidean number space E^n , containing a neighborhood of infinity and having a smooth internal boundary $F\mathcal{E}$. A continuous function $\phi(x)$ is prescribed on $F\mathcal{E}$. We wish then to find a function $u = u(x)$ which is of