

# *A Martingale Representation Theorem*

J. L. SNELL & R. E. WILLIAMSON<sup>1</sup>

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In recent years martingale theory has been applied to the study of a variety of stochastic processes by use of the following technique: Let  $\{X_t\}$  be a stochastic process. Let  $f(x, t)$  be a function of  $x$  and  $t$  such that  $\{f(X_t, t)\}$  is a martingale, that is, such that

$$E \{f(X_t, t) \mid X_u, u \leq s\} = f(X_s, s)$$

with probability one. We say that  $f(x, t)$  is a *martingale function* for  $\{X_t\}$ . If  $f(x, t)$  is non-negative, the convergence theorem for discrete time non-negative martingales and the theorems concerning continuity properties of sample functions for continuous time martingales may then be exploited to obtain information about the sample functions of the original process. This technique has been employed by DOOB in a sequence of papers relating Markov processes and potential theory. See DOOB [1]. The first use of the technique was made by VILLE [7] to obtain a proof of one part of the law of the iterated logarithm for Bernoulli trials. His procedure was the following: Let  $S_n$  be the number of successes in  $n$  trials with probability  $p$  for success. He wished to prove that if  $\{a_n\}$  is a sequence tending to infinity faster than

$$\{np + \lambda \sqrt{2npq \log \log n}\}, \quad \lambda > 1,$$

then

$$\Pr [S_n = a_n \text{ infinitely often}] = 0.$$

To do this he first verified that any function of the form

$$(1) \quad f(x, n) = \int_0^1 \left(\frac{s}{p}\right)^x \left(\frac{1-s}{1-p}\right)^{n-x} dF(s)$$

is a martingale function for the process  $\{S_n\}$ . He showed, in fact, that every martingale function for this process may be represented in the form (1). He then showed that if  $\{a_n\}$  tends to infinity faster than

$$\{np + \lambda \sqrt{2npq \log \log n}\}, \quad \lambda > 1,$$

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