

Whitehead Products and Projective Spaces

M. G. BARRATT, I. M. JAMES & N. STEIN

Communicated by E. SPANIER

It has long been known how to express the homotopy groups of the classical projective spaces in terms of the homotopy groups of spheres. The object of this note is to provide expressions for the Whitehead products in similar terms. In particular, all Whitehead products are zero in the complex projective 3-space, although this is not an H -space.

1. The Samelson product. Let G be a topological group. Associated with a principal G -bundle over a space X is a transgression operator

$$\Delta : \pi_{r+1}(X) \rightarrow \pi_r(G), \quad r \geq 0.$$

Let $p, q \geq 1$ and let $[\xi, \eta] \in \pi_{p+q+1}(X)$ be the Whitehead product of $\xi \in \pi_{p+1}(X)$, $\eta \in \pi_{q+1}(X)$. We prove that

$$(1.1) \quad (-1)^p \Delta[\xi, \eta] = \langle \Delta\xi, \Delta\eta \rangle,$$

where the right hand side denotes the Samelson¹ product, defined, as in §1 of [15], by the use of commutators.

Instead of following [12] we prove (1.1) by a combination of results of Fox ([5]) and G. W. WHITEHEAD ([15]). Let Y^A denote the function space of base-point-preserving maps of a complex A into a space Y . We give G^A the group structure inherited from G . From the given G -bundle B over X we obtain a G^A -bundle B^A over X^A and a corresponding transgression homomorphism

$$\Delta^A : \pi_1(X^A) \rightarrow \pi_0(G^A).$$

When $A = S^r$, we can identify

$$\pi_1(X^A) = \pi_{r+1}(X), \quad \pi_0(G^A) = \pi_r(G),$$

so that Δ^A is transformed into Δ . If $A = B \times C$, we embed Y^B, Y^C in Y^A in the usual way ([2]).

Take $A = S^p \times S^q$; we can regard S^{p+q} as obtained from A by pinching

¹ The first mention of such a product appears to be in [12].