

Solutions to Riccati's Problem as Functions of Initial Values

G. S. McCARTY, JR.

Communicated by E. HILLE

Introduction. R. M. REDHEFFER has constructed [1] a set of functions defined by the solution of a RICCATI problem. Considering the solutions of his defining equations as functions of both the independent variable y and an initial point x at which the functions vanish, he shows that if

$$(1) \quad \begin{aligned} u_v &= a(y) + 2b(y)u + c(y)u^2, & u(x, x) &= 0; \\ v_v &= b(y) + c(y)u, & v(x, x) &= 0; \\ w_v &= c(y) \exp(2v), & w(x, x) &= 0 \end{aligned}$$

in an open convex set symmetric about $y = x$; and if $v(x, y)$ is continuous in x , then the partial derivatives with respect to x exist there and satisfy

$$(2) \quad \begin{aligned} -w_x &= c(x) + 2b(x)w + a(x)w^2; \\ -v_x &= b(x) + a(x)w; \\ -u_x &= a(x) \exp(2v). \end{aligned}$$

Furthermore, REDHEFFER shows the following functional equations are valid whenever u, v and w satisfy (1) and the points (x, y) , (x, z) and (y, z) are in the set:

$$(3) \quad \begin{aligned} \frac{u(x, z) - u(y, z)}{u(x, y)} &= \exp[-v(x, y) + v(x, z) + v(y, z)], \\ 1 - u(x, y)w(y, z) &= \exp[v(x, y) - v(x, z) + v(y, z)], \\ \frac{w(x, z) - w(x, y)}{w(y, z)} &= \exp[v(x, y) + v(x, z) - v(y, z)]. \end{aligned}$$

In Theorem III of [1] the first of equations (3) is considered in a form which holds when $u(x, x)$ and $v(x, x)$ differ from zero, namely

$$\frac{u(x, z) - u(y, z)}{u(x, y) - u(y, y)} = \frac{\exp[v(x, z) + v(y, z)]}{\exp[v(x, y) + v(y, y)]}$$