

On Reynolds Operators in Finite-Dimensional Algebras

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1. **Definitions.** Let \mathfrak{a} be a commutative algebra with identity e over a field F of characteristic not equal to two. A REYNOLDS operator in \mathfrak{a} is an everywhere defined linear operator of \mathfrak{a} into itself satisfying the REYNOLDS identity (after the British engineer OSBORNE REYNOLDS)

$$(1) \quad R(fg) = (Rf)(Rg) + R[(f - Rf)(g - Rg)], \quad f, g \in \mathfrak{a}.$$

The origin of this identity, as well as its numerous applications to operator theory, probability and turbulence theory, will be discussed elsewhere. The reader is referred to the works of Mme. MARIE-LOUISE DUBREIL-JACOTIN (cf. [6], [7] and [8]) for recent surveys of the literature related to REYNOLDS operators.

A related concept is that of a *smoothing operator*; such an operator A is an everywhere defined linear operator in the algebra \mathfrak{a} which satisfies the identity (the so-called averaging identity)

$$(2) \quad A[f(Ag)] = (Af)(Ag), \quad f, g \in \mathfrak{a}.$$

In a lattice-ordered algebra (cf. [4]) an order-preserving smoothing operator is called an *averaging operator*. For a survey of the theory of averaging operators, see [3]; the relationship with the notion of conditional expectation in probability theory is discussed in [12]. Order and lattice properties will not be considered in this work. Smoothing operators have been studied rather completely, and can be considered as known (cf. [3]). Smoothing operators in the algebra of continuous functions $C(S)$ over a compact Hausdorff space have been completely classified (cf. [2], [3] and [10]). Averaging operators in the algebra of measurable functions have also been studied (cf. [12]).

The following counter-example, due to MOLINARO (cf. [6]), shows that in general the REYNOLDS identity does not imply the averaging identity. Let \mathfrak{a}

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